COALGEBRAS OF WORDS AND PHRASES

VLADIMIR TURAEV

Abstract. We introduce two constructions of a coassociative comultiplication in the algebra of phrases in a given alphabet. As a preliminary step we give two constructions of a pre-Lie comultiplication in the module generated by words.

1. Introduction

We give two new constructions of non-commutative non-cocommutative Hopf algebras (of infinite rank). These constructions can be summarized as follows. A module M over a commutative ring gives rise to the tensor algebra $\mathcal{W} = T(M) = \oplus_{m\geq 0} M^{\otimes m}$ and to its positive subalgebra $\mathcal{V} = T_+(M) = \oplus_{m\geq 1} M^{\otimes m}$. Under certain additional assumptions, we define comultiplications in the tensor algebras $\mathcal{P} = T(\mathcal{V})$ and $\mathcal{Q} = T(\mathcal{W})$ so that they become Hopf algebras.

A comultiplication Δ in a graded algebra $\oplus_{m\geq 0} T^m$ has a leading term which is the homomorphism $T^1 \to T^1 \otimes T^1$ obtained by applying Δ and then projecting to $T^1 \otimes T^1$. This leading term yields the first approximation to Δ and is often interesting in itself. The leading terms of our comultiplications in $\mathcal{P}$ and $\mathcal{Q}$ are homomorphisms $\mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ and $\mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$. They turn out to be pre-Lie comultiplications in the sense explained below. In particular, dualizing and skew-symmetrizing them, we obtain Lie brackets in the dual modules $\mathcal{V}^*$ and $\mathcal{W}^*$.

The study of tensor algebras can be reformulated in terms of words and phrases. Suppose from now on that the module M is free with basis $\mathcal{A}$. Consider the basis of $\mathcal{V} = T_+(M)$ formed by the vectors $A_1 \otimes A_2 \otimes ... \otimes A_m$ where $m \geq 1$ and $A_1, A_2, ..., A_m \in \mathcal{A}$. Omitting the symbol $\otimes$ we can identify these vectors with words in the alphabet $\mathcal{A}$. A basis in $\mathcal{P} = T(\mathcal{V})$ is formed by finite sequences of basis vectors in $\mathcal{V}$, that is by sequences of words. We call such sequences *phrases*. Natural bases in $\mathcal{W} = T(M)$ and $\mathcal{Q} = T(\mathcal{W})$ are similarly interpreted as words and phrases with the difference that here we allow an empty word. This interpretation gives a pleasant linguistical flavour to the theory. It places the study of words and phrases in the setting of Hopf algebras, Lie algebras, and related algebraic objects.

One well known construction of a comultiplication in $\mathcal{W}$ is based on the notion of a subword. For us, a *word* is a finite sequence of elements of $\mathcal{A}$ and a *subword* is any subsequence. Given a word w we can form the sum $\sum w' \otimes w/w'$ where w' runs over all subwords of w and the word w/w' is obtained by deleting w' from w. This yields the familiar *shuffle comultiplication* in $\mathcal{W}$. It is coassociative and cocommutative.

Our comultiplications are based on two different constructions. They are determined by certain additional data which should be fixed from the very beginning. The comultiplications in $\mathcal{V}$ and $\mathcal{P} = T(\mathcal{V})$ depend on a choice of a so-called stable set of words L in the alphabet $\mathcal{A}$. We associate with a word w the sum $\sum w' \otimes w/w'$ where w' runs over all subwords of w formed by consecutive letters and belonging to L. This yields a pre-Lie comultiplication ρ_L in $\mathcal{V}$. In a similar way we construct a coassociative comultiplication in $\mathcal{P}$ with leading term ρ_L.

The second construction begins with fixing a mapping μ from $\mathcal{A} \times \mathcal{A}$ to the ground ring. The corresponding pre-Lie comultiplication ρ_μ in $\mathcal{W}$ is defined as follows. For a subword of length two $a = AB$ of a word w, set $\mu_a = \mu(A, B)$ and denote by w'_a be the subword of w formed by the letters of w appearing between A and B. Let w''_a be the word obtained by deleting both a and w'_a from w. Then $\rho_\mu(w) = \sum_a \mu_a w'_a \otimes w''_a$. In a similar way we construct a coassociative comultiplication in $\mathcal{Q} = T(\mathcal{W})$ with leading term ρ_μ.

Under a certain choice of μ, the latter comultiplication is closely related to the Connes-Kreimer comultiplication in the algebra of rooted trees, see [CK], or, more precisely, to its non-commutative version for planar rooted trees due to Foissy [Fo].

Although we can directly define our comultiplications in $\mathcal{P}$ and $\mathcal{Q}$, we begin with a study of their leading terms. In Section 2 we discuss relevant notions from the theory of pre-Lie multiplications and comultiplications. Sections 3 and 4 are concerned with the comultiplication in $\mathcal{P}$ derived from a stable set of words:

the leading term is defined in Section 3 and the comultiplication itself is defined in Section 4. Sections 5 and 6 are concerned with the comultiplication in $\mathcal{Q}$ derived from a mapping μ as above: the leading term is defined in Section 5 and the comultiplication itself is defined in Section 6. At the end of Section 6 we discuss connections with the theory of planar rooted trees.

Throughout the paper, we fix a commutative ring with unit R. The symbol $\otimes$ denotes the tensor product of R-modules over R.

2. Pre-Lie coalgebras and left-handed bialgebras

2.1. **Pre-Lie algebras.** By a *multiplication* in an R-module M we mean an R-bilinear mapping $M\times M \to M$. A (left) *pre-Lie algebra* over R is an R-module M endowed with a multiplication $M \times M \to M$, denoted $\star$, such that for any $f, g, h \in M$,

$$f \star (g \star h) - (f \star g) \star h = g \star (f \star h) - (g \star f) \star h. \tag{2.1.1}$$

The mapping $\star$ is called then a *pre-Lie multiplication*, see [Ge] and [Vi]. For example, any associative multiplication is pre-Lie. A fundamental property of a pre-Lie multiplication $\star$ in M is that the formula $[f, g] = f \star g - g \star f$ defines a Lie bracket in M. The Jacobi identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

for $f, g, h \in M$ is a direct consequence of (2.1.1). In this way every pre-Lie algebra becomes a Lie algebra.

There is a similar notion of right pre-Lie algebras. We will consider only left pre-Lie algebras and refer to them simply as pre-Lie algebras. Non-trivial examples of pre-Lie algebras can be obtained from derivations in algebras. A *derivation* in an associative R-algebra M is an R-linear homomorphism $d : M \to M$ such that $d(fg) = d(f)g + f\, d(g)$ for any $f, g \in M$. An easy computation shows that for any $a, b \in M$, the formula $f \star g = afg + bfd(g)$ defines a pre-Lie multiplication in M.

2.2. **Pre-Lie coalgebras.** A *comultiplication* in an R-module A is an R-linear homomorphism $\rho : A \to A\otimes A$. We associate with such ρ a homomorphism $\tilde\rho : A \to A \otimes A \otimes A$ sending $a \in A$ to

$$\tilde\rho(a) = (\mathrm{id}_A \otimes\rho)\rho(a) - (\rho \otimes \mathrm{id}_A)\rho(a).$$

The comultiplication ρ is *coassociative* if $\tilde\rho = 0$.

Dualizing Formula 2.1.1, we obtain a notion of a pre-Lie coalgebra. Namely, a *pre-Lie coalgebra* is an R-module A endowed with a comultiplication $\rho : A \to A^{\otimes 2} = A \otimes A$ such that for all $a \in A$,

$$P^{1,2}\, \tilde\rho(a) = \tilde\rho(a). \tag{2.2.1}$$

Here and below given a module U, we denote by $P^{1,2}$ the endomorphism of $A^{\otimes 2} \otimes U$ permuting the first two tensor factors, i.e., mapping $a \otimes b \otimes u$ to $b \otimes a \otimes u$ for $a, b \in A, u \in U$. A comultiplication ρ satisfying (2.2.1) is a *pre-Lie comultiplication*. Clearly, a coassociative comultiplication is pre-Lie.

Given a pre-Lie coalgebra (A, ρ), consider the module $A^* = \mathrm{Hom}_R(A, R)$ and the evaluation pairing $\langle\, ,\, \rangle : A \otimes A^* \to R$. The comultiplication ρ induces a pre-Lie multiplication $\star_\rho$ in A^* by

$$\langle a, f \star_\rho g\rangle = \sum_i \langle a_i', f\rangle\, \langle a_i'', g\rangle \in R \tag{2.2.2}$$

for any $f, g \in A^*$, $a \in A$ and any (finite) expansion $\rho(a) = \sum_i a_i' \otimes a_i'' \in A^{\otimes 2}$. In this way A^* becomes a pre-Lie algebra.

2.3. **Comodules over pre-Lie coalgebras.** A *(left) comodule* over a pre-Lie coalgebra (A, ρ) is a pair (an R-module U, an R-homomorphism $\theta : U \to A \otimes U$) such that $\tilde\theta = (\mathrm{id}_A \otimes\theta)\theta - (\rho \otimes \mathrm{id}_U)\theta : U \to A^{\otimes 2} \otimes U$ satisfies $P^{1,2}\tilde\theta = \tilde\theta$. Given a comodule (U, θ) over (A, ρ), we define a right action of A^* on U by

$$uf = \sum_i \langle u_i', f\rangle\, u_i''$$

for any $f \in A^*$, $u \in U$, and any expansion $\theta(u) = \sum_i u_i' \otimes u_i''$ with $u_i' \in A, u_i'' \in U$.

Lemma 2.3.1. *The action of A^* on U is a Lie algebra action.*

Proof. We must prove that $u[f,g]=(uf)g-(ug)f$ for all $f,g\in A^*$, $u\in U$ where $[f,g]=f\star_\rho g-g\star_\rho f$. We shall use Sweedler's notation for the expansion $\theta(u)=\sum_i u'_i\otimes u''_i$ and write it in the form $\theta(u)=\sum_{(u)}u'\otimes u''$. Similarly, for $a\in A$, we write $\rho(a)=\sum_{(a)}a'\otimes a''$. By definition,

$$(uf)g=\sum_{(u)}\langle u',f\rangle u''g=\sum_{(u),(u'')}\langle u',f\rangle\langle (u'')',g\rangle (u'')''.$$

Hence

$$(uf)g-(ug)f=\sum_{(u),(u'')}\langle u',f\rangle\langle (u'')',g\rangle (u'')''-\langle (u'')',f\rangle\langle u',g\rangle (u'')''.$$

To compute $u[f,g]$, observe that for $a\in A$,

$$\langle a,[f,g]\rangle=\langle a,f\star_\rho g-g\star_\rho f\rangle=\sum_{(a)}\langle a',f\rangle\,\langle a'',g\rangle-\langle a'',f\rangle\,\langle a',g\rangle.$$

Therefore

$$u[f,g]=\sum_{(u)}\langle u',[f,g]\rangle\,u''=\sum_{(u),(u')}\langle (u')',f\rangle\,\langle (u')'',g\rangle\,u''-\langle (u')'',f\rangle\,\langle (u')',g\rangle\,u''.$$

The equality $u[f,g]=(uf)g-(ug)f$ follows now from the formula

$$\sum_{(u),(u')}(u')'\otimes(u')''\otimes u''-(u')''\otimes(u')'\otimes u''=\sum_{(u),(u'')}u'\otimes(u'')'\otimes(u'')''-(u'')'\otimes u'\otimes(u'')''$$

which is a reformulation of the equality $P^{1,2}\tilde\theta=\tilde\theta$. □

Lemma 2.3.1 implies that the formula $fu=-uf$ with $u\in U,f\in A^*$ defines a left Lie algebra action of A^* on U.

It is clear that $(A,\theta=\rho:A\to A\otimes A)$ is a comodule over (A,ρ). Lemma 2.3.1 yields a Lie algebra action of A^* on A.

2.4. **Left-handed bialgebras.** By a *bialgebra* we shall mean a pair (T,Δ) where T is an associative unital R-algebra and $\Delta:T\to T\otimes T$ is a coassociative algebra comultiplication in T. The words "algebra comultiplication" mean that $\Delta(1)=1\otimes 1$ and $\Delta(ab)=\Delta(a)\Delta(b)$ for any $a,b\in T$. Here multiplication in $T\otimes T$ is defined by $(a\otimes a')(b\otimes b')=ab\otimes a'b'$ for $a,b,a',b'\in T$. We do not require the existence of a counit $T\to R$ although in all our constructions of bialgebras there will be a counit.

The comultiplication in a bialgebra (T,Δ) induces an associative multiplication in $T^*=\mathrm{Hom}_R(T,R)$ by $fg(a)=\sum_{(a)}f(a')\,g(a'')$ for $a\in T,f,g\in T^*$, and any expansion $\Delta(a)=\sum_{(a)}a'\otimes a''$. This makes T^* into an associative algebra. If T has a counit, then T^* is a unital algebra. Dualizing multiplication in T we obtain a homomorphism from T^* to a certain completion of $T^*\otimes T^*$. We call this homomorphism *quasi-comultiplication.*

A bialgebra (T,Δ) is *graded* if T splits as a direct sum of submodules $T=\oplus_{m\geq 0}T^m$ such that $1\in T^0$ and $T^mT^n\subset T^{m+n}$ for all m,n. Set $T_+=\oplus_{m\geq 1}T^m\subset T$. A graded bialgebra (T,Δ) is *left-handed* if for any $a\in T^1$,

$$\Delta(a)-a\otimes 1-1\otimes a\in T_+\otimes T^1.$$

The *leading term* of Δ is then the homomorphism $(\pi\otimes\pi)\Delta|_{T^1}:T^1\to T^1\otimes T^1$ where $\pi:T\to T^1$ is the projection. We shall mainly apply these definitions in the case where

$$T=T(A)=\oplus_{m\geq 0}A^{\otimes m}$$

is the tensor algebra of an R-module A. Here $A^{\otimes 0}=R$, $A^{\otimes 1}=A$, and $A^{\otimes m}$ is the tensor product of m copies of A for $m\geq 2$.

The next lemma relates left-handed bialgebras with pre-Lie coalgebras.

Lemma 2.4.1. *Let $(T=\oplus_{m\geq 0}T^m,\Delta)$ be a left-handed graded bialgebra such that T^1 generates $T_+=\oplus_{m\geq 1}T^m$. Then the leading term $\rho:T^1\to T^1\otimes T^1$ of Δ is a pre-Lie comultiplication in T^1.*

Proof. For $a\in T^1$, we can expand

$$\Delta(a)=a\otimes 1+1\otimes a+\sum_i a'_i\otimes a''_i+\sum_j a^1_ja^2_j\otimes a^3_j\ (\mathrm{mod}\,\oplus_{m\geq 3}T^m\otimes T^1)$$

where i, j run over finite sets of indices and $a'_i, a''_i, a^1_j, a^2_j, a^3_j \in T^1$. Clearly, $\rho(a) = \sum_i a'_i \otimes a''_i$. Our assumptions imply that $\Delta(T^m) \subset \oplus_{k+n\geq m} T^k \otimes T^n$ for all m. Computing $(\mathrm{id} \otimes \Delta)\Delta(a)$ and $(\Delta \otimes \mathrm{id})\Delta(a)$ modulo $\oplus_{k,n,r,k+n+r\geq 4} T^k \otimes T^n \otimes T^r$ and equating the resulting expressions we obtain that

$$\tilde{\rho}(a) = \sum_j a^1_j \otimes a^2_j \otimes a^3_j + a^2_j \otimes a^1_j \otimes a^3_j.$$

Therefore $P^{1,2}\tilde{\rho}(a) = \tilde{\rho}(a)$. $\square$

Let (T, Δ) be as in the lemma. A *(left) comodule* over (T, Δ) is a pair (an R-module U, an R-homomorphism $\Theta : U \to T \otimes U$) such that $(\Delta \otimes \mathrm{id}_U)\Theta = (\mathrm{id}_T \otimes \Theta)\Theta$ and $\Theta(u) - 1 \otimes u \in T_+ \otimes U$ for any $u \in U$. Then U becomes a right module over the algebra T^* by $uf = \sum_{(u)} f(u')u''$ for $f \in T^*$, $u \in U$, and any finite expansion $\Theta(u) = \sum_{(u)} u' \otimes u''$. For example, set $U = T^1$ and define $\Theta : U \to T \otimes U$ by $\Theta(u) = \Delta(u) - u \otimes 1$ for $u \in U$. The pair (U, Θ) is a comodule over (T, Δ) (cf. the proof of Theorem 4.2.1 below). This makes $U = T^1$ into a right T^*-module.

2.5. **Remarks.** 1. Lemma 2.4.1 suggests to study when a given pre-Lie comultiplication is induced from a left-handed comultiplication in a graded bialgebra. For the pre-Lie comultiplications defined in the next sections this will be always the case.

2. Consider a pre-Lie coalgebra (A, ρ) and a submodule $B \subset A$ such that $\rho(A) \subset B \otimes A$ and $\rho(B) \subset B \otimes B$. Then $(B, \rho|_B)$ is a pre-Lie coalgebra and $(A, \theta = \rho : A \to B \otimes A)$ is a comodule over $(B, \rho|_B)$. The associated action of B^* on A is compatible with the action of A^* on A via the Lie algebra homomorphism $A^* \to B^*$ induced by the inclusion $B \subset A$.

3. For a pre-Lie coalgebra (A, ρ), the homomorphism $\rho - P^{1,2}\rho : A \to A^{\otimes 2}$ is a Lie cobracket (cf., for instance [Tu]). In particular in Lemma 2.4.1, $\rho - P^{1,2}\rho$ is a Lie cobracket in T^1. This still holds if the left-handedness assumption on Δ is weakened to $\Delta(a) - a \otimes 1 - 1 \otimes a \in T_+ \otimes T_+$ for all $a \in T^1$.

4. Lemma 2.4.1 extends to comodules as follows. Let (T, Δ) be as in this lemma. The leading term of a comodule (U, Θ) over (T, Δ) is the homomorphism $\theta = (\pi \otimes \mathrm{id})\Theta : U \to T^1 \otimes U$ where $\pi : T \to T^1$ is the projection. A direct computation shows that (U, θ) is a comodule over the pre-Lie coalgebra $(T^1, \rho = (\pi \otimes \pi)\Delta|_{T^1})$. Note also that if $T^0 = R$, then the projection $T \to T^0 = R$ is a counit of (T, Δ).

3. Coalgebra of words

3.1. **Words.** For us, an *alphabet* is an arbitrary set and *letters* are its elements. Throughout the paper we fix an alphabet $\mathcal{A}$. A *word of length* $m \geq 1$ is a mapping from the set $\{1, 2, ..., m\}$ to $\mathcal{A}$. By definition, there is a unique word of length 0 called the *empty word* and denoted ϕ. To present a word $w : \{1, 2, ..., m\} \to \mathcal{A}$ it is enough to write down the sequence of letters $w(1)w(2)...w(m)$. For instance, the symbol ABA represents the word $\{1, 2, 3\} \to \mathcal{A}$ sending 1 and 3 to $A \in \mathcal{A}$ and sending 2 to $B \in \mathcal{A}$.

Writing down consecutively the letters of two words w and x we obtain their concatenation wx. For instance, the concatenation of $w = ABA$ and $x = BB$ is the word $wx = ABABB$.

A word x is a *factor* of a word w if $w = yxz$ for certain words y, z. A factor x of w is *proper* if $x \neq w$.

Given a word w of length $m \geq 1$ and numbers $1 \leq i \leq j \leq m+1$, set $w_{i,j} = w(i)w(i+1)...w(j-1)$. This word of length $j - i$ is a factor of $w = w_{1,i}w_{i,j}w_{j,m+1}$. The word $w_{i,j}$ is empty iff $i = j$ and proper iff $(i, j) \neq (1, m+1)$.

Let $\mathcal{W} = \mathcal{W}(\mathcal{A})$ be the free R-module freely generated by the set of words in the alphabet $\mathcal{A}$. A typical element of $\mathcal{W}$ is a finite formal linear combination of words with coefficients in R. Each word w represents a vector in $\mathcal{W}$ denoted also w. These vectors form a basis of $\mathcal{W}$. Let $\mathcal{V} = \mathcal{V}(\mathcal{A})$ be the submodule of $\mathcal{W}$ generated by non-empty words. Clearly, $\mathcal{W} = \mathcal{V} \oplus R\phi$.

3.2. **Stable sets of words.** A set L of non-empty words is *stable* if it satisfies the following condition:

$(*)$ For any word w of length $m \geq 1$ and any indices $1 \leq i < j \leq m+1$ such that $(i, j) \neq (1, m+1)$ and $w_{i,j} \in L$, the word w belongs to L if and only if the word $w_{1,i}w_{j,m+1}$ belongs to L.

The words belonging to a stable set L will be called *L-words*. The "only if" part of $(*)$ means that striking out from any L-word a proper factor belonging to L one obtains an L-word. The "if" part of $(*)$ means that inserting an L-word in an L-word one obtains an L-word. In particular, concatenation of L-words is an L-word.

We give a few examples of stable sets of words.

(1) The set of all non-empty words and the void set of words are stable.

To give the next example, denote by $f_A(w)$ the number of appearances of a letter $A \in \mathcal{A}$ in a word w. For instance, $f_A(ABA) = 2$.

(2) Pick $A \in \mathcal{A}$. The set of non-empty words w such that $f_A(w) = 0$ is stable. Similarly, for any integer N, the set of non-empty words w such that $f_A(w)$ is divisible by N is stable.

(3) Pick letters $A_1, ..., A_n \in \mathcal{A}$ and elements $g_1, ..., g_n$ of a certain abelian group. The set of non-empty words w such that $f_{A_1}(w)g_1 + ... + f_{A_n}(w)g_n = 0$ is stable.

Observe finally that the intersection of any family of stable sets of words is stable. A union of stable sets can be non-stable.

3.3. **Pre-Lie coalgebra of words.** We fix a stable set of words L and derive from it a pre-Lie comultiplication ρ_L in the module $\mathcal{V}$.

A *simple cut* of a word w of length $m \geq 1$ is a pair of indices $1 \leq i < j \leq m+1$ such that $(i,j) \neq (1, m+1)$ and $w_{i,j} \in L$. To indicate that (i,j) is a simple cut of w we write $(i,j) \prec w$. Set

$$\rho_L(w) = \sum_{(i,j)\prec w} w_{i,j} \otimes w_{1,i} w_{j,m+1} \in \mathcal{V} \otimes \mathcal{V}$$

where the sum runs over all simple cuts of w. Note that the words $w_{i,j}$ and $w_{1,i}w_{j,m+1}$ are necessarily non-empty. This defines ρ_L on the basis of $\mathcal{V}$ and extends by linearity to a comultiplication $\rho_L : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$. For example, if $A, B \in \mathcal{A}$ and L is the set of all non-empty words, then $\rho_L(A) = 0$, $\rho_L(AB) = A \otimes B + B \otimes A$,

$$\rho_L(ABA) = A \otimes BA + B \otimes AA + A \otimes AB + AB \otimes A + BA \otimes A.$$

Theorem 3.3.1. *ρ_L is a pre-Lie comultiplication in $\mathcal{V}$.*

Proof. For a word w of length $m \geq 1$,

$$(\mathrm{id} \otimes \rho_L)\rho_L(w) = \sum_{(i,j)\prec w} w_{i,j} \otimes \rho_L(w_{1,i}w_{j,m+1}).$$

To compute $w_{i,j} \otimes \rho_L(w_{1,i}w_{j,m+1})$, we need to consider factors of $w_{1,i}w_{j,m+1}$. There are three kinds of them: (i) the factors of $w_{1,i-1}$, (ii) the factors of $w_{j+1,m+1}$, and (iii) the factors obtained by concatenation of some $w_{i',i}$ with some $w_{j,j'}$ where $i' \leq i, j \leq j'$. Here and below $i, j, i', j' \in \{1, 2, ..., m\}$. The factors of $w_{1,i}w_{j,m+1}$ of type (i) contribute to $w_{i,j} \otimes \rho_L(w_{1,i}w_{j,m+1})$ the expression

$$x_{i,j} = \sum_{(i',j')\prec w, j'<i} w_{i,j} \otimes w_{i',j'} \otimes w_{1,i'}w_{j',i}w_{j,m+1}.$$

The factors of $w_{1,i}w_{j,m+1}$ of type (ii) contribute

$$y_{i,j} = \sum_{(i',j')\prec w, j<i'} w_{i,j} \otimes w_{i',j'} \otimes w_{1,i}w_{j,i'}w_{j',m+1}.$$

The factors of $w_{1,i}w_{j,m+1}$ of type (iii) contribute

$$z_{i,j} = \sum_{i'\leq i, j\leq j', (i',j')\neq(1,m+1), w_{i',i}w_{j,j'}\in L} w_{i,j} \otimes w_{i',i}w_{j,j'} \otimes w_{1,i'}w_{j',m+1}.$$

Recall that $w_{i,j} \in L$. By definition of a stable set of words, $w_{i',i}w_{j,j'} \in L$ iff $(i',j') \neq (i,j)$ and $w_{i',j'} \in L$. Therefore the conditions on i', j' in the latter sum are equivalent to the conditions $i' \leq i, j \leq j', (i',j') \prec w, (i',j') \neq (i,j)$. We obtain

$$\sum_{(i,j)\prec w} z_{i,j} = \sum_{(i,j)\prec w, (i',j')\prec w, i'\leq i, j\leq j', (i',j')\neq(i,j)} w_{i,j} \otimes w_{i',i}w_{j,j'} \otimes w_{1,i'}w_{j',m+1}$$

$$= \sum_{(i',j')\prec w} \rho_L(w_{i',j'}) \otimes w_{1,i'}w_{j',m+1} = (\rho_L \otimes \mathrm{id})\rho_L(w).$$

Hence,

$$\tilde{\rho}_L(w) = (\mathrm{id} \otimes \rho_L)\rho_L(w) - (\rho_L \otimes \mathrm{id})\rho_L(w) = \sum_{(i,j)\prec w} x_{i,j} + y_{i,j}.$$

It remains to observe that

$$P^{1,2}\Big(\sum_{(i,j)\prec w} x_{i,j}\Big) = \sum_{(i,j)\prec w} y_{i,j}.$$

Therefore $\tilde{\rho}_L(w)$ is invariant under $P^{1,2}$. $\square$

3.4. **Word indicators.** The elements of the module $\mathcal{V}^* = \mathrm{Hom}_R(\mathcal{V}, R)$ are called *word indicators*. The module $\mathcal{V}^*$ admits a decreasing filtration $\mathcal{V}^* = \mathcal{V}^{*(1)} \supset \mathcal{V}^{*(2)} \supset ...$ where $\mathcal{V}^{*(m)}$ consists of the word indicators annihilating all non-empty words of length $\leq m-1$. Clearly, $\cap_m \mathcal{V}^{*(m)} = 0$. We can consider infinite sums in $\mathcal{V}^*$ as follows. Let $f_1, f_2, \ldots \in \mathcal{V}^*$ be a sequence of word indicators such that for any $m \geq 1$ all terms of the sequence starting from a certain place belong to $\mathcal{V}^{*(m)}$. Then for any $a \in \mathcal{V}$, the sum $f(a) = f_1(a) + f_2(a) + \cdots$ contains only a finite number of non-zero terms and the formula $a \mapsto f(a) : \mathcal{V} \to R$ defines a word indicator $f = f_1 + f_2 + \cdots$. A similar construction shows that the natural homomorphism $\mathcal{V}^* \to \mathrm{proj\,lim}_m(\mathcal{V}^*/\mathcal{V}^{*(m)})$ is an isomorphism.

By the general theory, the pre-Lie comultiplication ρ_L in $\mathcal{V}$ induces a pre-Lie multiplication $\star_L$ and a Lie bracket $[f,g]_L = f \star_L g - g \star_L f$ in $\mathcal{V}^*$. It is clear that $\mathcal{V}^{*(m)} \star_L \mathcal{V}^{*(n)} \subset \mathcal{V}^{*(m+n)}$ and $[\mathcal{V}^{*(m)}, \mathcal{V}^{*(n)}]_L \subset \mathcal{V}^{*(m+n)}$ for all m, n. Therefore $\mathcal{V}^* = \mathrm{proj\,lim}_m(\mathcal{V}^*/\mathcal{V}^{*(m)})$ is a projective limit of nilpotent Lie algebras.

Recall the Lie algebra action of $\mathcal{V}^*$ on $\mathcal{V}$ induced by ρ_L. For $f \in \mathcal{V}^*$ and a word w of length $m \geq 1$,

$$fw = -wf = -\sum_{(i,j) \prec w} \langle w_{i,j}, f \rangle \, w_{1,i} w_{j,m+1}.$$

All word indicators annihilating L lie in the kernel of this action.

The module $\mathcal{V}$ has an increasing filtration $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset ...$ where $\mathcal{V}_m$ is generated by non-empty words of length $\leq m$. It is clear that $f\mathcal{V}_m \subset \mathcal{V}_{m-1}$ for all m and $f \in \mathcal{V}^*$. Thus the action of $\mathcal{V}^*$ on $\mathcal{V}_m$ is nilpotent for all m.

The restriction of $\rho_L : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ to the module $RL \subset \mathcal{V}$ generated by L is a pre-Lie comultiplication in RL since $\rho_L(RL) \subset RL \otimes RL$. Clearly $\rho_L(\mathcal{V}) \subset RL \otimes \mathcal{V}$ so that $\mathcal{V}$ becomes a comodule over the pre-Lie coalgebra RL. The associated actions of $(RL)^*$ and $\mathcal{V}^*$ on $\mathcal{V}$ are compatible via the Lie algebra homomorphism $\mathcal{V}^* \to (RL)^*$ induced by the inclusion $RL \subset \mathcal{V}$. This is a special case of Remark 2.5.2.

3.5. **Example.** Let the alphabet $\mathcal{A}$ consist of only one letter A. Denote by A^m the word consisting of m letters A where $m \geq 1$. A stable set of words L is determined by a positive integer N and consists of all words A^m with m divisible by N. The comultiplication ρ_L is computed by

$$\rho_L(A^m) = \sum_{k \geq 1, kN < m} (m+1-kN) A^{kN} \otimes A^{m-kN}$$

for any $m \geq 1$. We can identify a word indicator f with its generating function $\sum_{m \geq 1} f(A^m) t^m$. One easily computes that $f \star_L g = (f\,|_N) \cdot (g + tg')$ where for a formal power series $f = \sum_{m \geq 1} a_m t^m$, we set $f\,|_N = \sum_{m \geq 1} a_{mN} t^{mN}$ and $f' = \sum_{m \geq 1} m a_m t^{m-1}$. For $N = 1$, we obtain $f \star_L g = fg + tfg'$.

3.6. **The group** $\mathrm{Exp}_L \mathcal{V}^*$**.** Assume in this subsection that $R \supset \mathbb{Q}$. For $f, g \in \mathcal{V}^*$, set

$$fg = f + g + \frac{1}{2}[f,g]_L + \frac{1}{12}([f,[f,g]_L]_L + [g,[g,f]_L]_L) + \cdots \in \mathcal{V}^*$$

where the right-hand side is the Campbell-Baker-Hausdorff series for $\log(e^f e^g)$. The resulting mapping $\mathcal{V}^* \times \mathcal{V}^* \to \mathcal{V}^*$ is a group multiplication in $\mathcal{V}^*$. Here $f^{-1} = -f$ and $0 \in \mathcal{V}^*$ is the group unit. The resulting group is denoted $\mathrm{Exp}_L \mathcal{V}^*$. Heuristically, this is the “Lie group” with Lie algebra $(\mathcal{V}^*, [\,,\,]_L)$. The equality $\mathcal{V}^* = \mathrm{proj\,lim}_m(\mathcal{V}^*/\mathcal{V}^{*(m)})$ implies that $\mathrm{Exp}_L \mathcal{V}^*$ is pro-nilpotent.

The action of $\mathcal{V}^*$ on $\mathcal{V}$ induced by ρ_L integrates to a group action of $\mathrm{Exp}_L \mathcal{V}^*$ on $\mathcal{V}$. To see this, denote by $\varphi(f)$ the additive endomorphism $a \mapsto fa$ of $\mathcal{V}$ determined by $f \in \mathcal{V}^*$. Set

$$e^{\varphi(f)}(a) = \sum_{k \geq 0} \frac{1}{k!} (\varphi(f))^k (a).$$

The latter sum makes sense since for any $a \in \mathcal{V}$, only a finite number of terms in the sum are non-zero. The formula $f \mapsto e^{\varphi(f)}$ defines a group homomorphism $\mathrm{Exp}_L \mathcal{V}^* \to \mathrm{End}(\mathcal{V})$, i.e., a group action of $\mathrm{Exp}_L \mathcal{V}^*$ on $\mathcal{V}$.

3.7. **Remarks.** 1. There is an embedding $\delta : \mathcal{V} \hookrightarrow \mathcal{V}^*$ mapping a word w into the word indicator δ_w whose value on w is 1 and whose value on all other words is 0. It is easy to check that the image of δ is closed under $\star_L$. This induces a pre-Lie multiplication $\circ_L$ in $\mathcal{V}$. For words $w = A_1A_2...A_m$ and $x = B_1B_2...B_n$ with $m, n \geq 1$, we have $w \circ_L x = 0$ if $w \notin L$ and

$$w \circ_L x = \delta^{-1}(\delta_w \star_L \delta_x) = \sum_{i=0}^{n} B_1...B_iA_1...A_mB_{i+1}...B_n$$

if $w \in L$. In the case where L is the set of all non-empty words, this pre-Lie multiplication in $\mathcal{V}$ is essentially due to Gerstenhaber [Ge].

2. With a slight modification, the notion of a stable set of words can be used to define a coassociative comultiplication in $\mathcal{W} = \mathcal{V} \oplus R\phi$. Let us say that a set of words S is *strongly stable* if $\phi \in S$ and for any word w and any its subword w' we have $w' \in S \Rightarrow (w \in S \Leftrightarrow w/w' \in S)$ where w/w' is the word obtained by deleting w' from w. Examples of strongly stable sets of words can be obtained by adjoining the empty word to the stable sets of words in the examples in Section 3.2. For a strongly stable set S and a word w, set $\Delta_S(w) = \sum_{w'} w' \otimes w/w'$ where w' runs over all subwords of w belonging to S. It is easy to see that Δ_S extends to a coassociative algebra comultiplication in $\mathcal{W}$. When S is the set of all words, this is the shuffle comultiplication mentioned in the introduction.

4. Hopf algebra of phrases

The results above suggest that there may exist a left-handed comultiplication in the tensor algebra $T(\mathcal{V})$ with leading term ρ_L. We construct such a comultiplication.

4.1. **Phrases.** A *phrase of length* $k \geq 1$ is a sequence of k words in the alphabet $\mathcal{A}$. Some (or all) of these words may be empty. We also allow an *empty phrase* consisting of 0 words and denoted 1. A more interesting example of a phrase: "*NIHIL NOVI SUB SOLE*" where $\mathcal{A}$ is the set of capital Latin letters. Here words are separated by blank spaces and the quotation marks indicate the beginning and the end of the phrase. These conventions, customary in ordinary texts, are not quite convenient for mathematical formulas. In the formulas we shall indicate the beginning and the end of a phrase by round brackets and separate words by vertical bars. Of course, we assume that the round brackets and the vertical bar are not letters in $\mathcal{A}$. The same example can be re-written as $(NIHIL \,|\, NOVI \,|\, SUB \,|\, SOLE)$. By abuse of notation, the phrase (w) consisting of one word w will be also denoted by w.

Let $\mathcal{Q} = \mathcal{Q}(\mathcal{A})$ be the free R-module freely generated by the set of phrases in the alphabet $\mathcal{A}$. Concatenation of phrases defines a multiplication in $\mathcal{Q}$ by

$$(w_1 \,|\, w_2 \,|\, ... \,|\, w_k)(x_1 \,|\, x_2 \,|\, ... \,|\, x_t) = (w_1 \,|\, w_2 \,|\, ... \,|\, w_k \,|\, x_1 \,|\, x_2 \,|\, ... \,|\, x_t)$$

where $k, t \geq 0$ and $w_1, ..., w_k, x_1, ..., x_t$ are words. This makes $\mathcal{Q}$ into an associative algebra with unit 1 (the empty phrase). This algebra is graded, the grading being the length of phrases.

In this section we shall focus on the subalgebra $\mathcal{P} = \mathcal{P}(\mathcal{A})$ of $\mathcal{Q}$ additively generated by phrases formed by non-empty words. Thus $(w_1 \,|\, w_2 \,|\, ... \,|\, w_k) \in \mathcal{P}$ iff $k = 0$ or $k \geq 1$ and all the words $w_1, ..., w_k$ are non-empty. Note that $1 \in \mathcal{P}$. The algebra $\mathcal{P}$ is graded: $\mathcal{P} = \mathcal{P}^0 \oplus \mathcal{P}^1 \oplus ...$ where $\mathcal{P}^k$ is the module additively generated by phrases of length k. The inclusion $\mathcal{V} = \mathcal{P}^1 \hookrightarrow \mathcal{P}$ extends to an algebra isomorphism $T(\mathcal{V}) \to \mathcal{P}$ and we identify the tensor algebra $T(\mathcal{V})$ with $\mathcal{P}$ along this isomorphism.

Phrases in the alphabet $\mathcal{A}$ can be viewed as words in the extended alphabet $\mathcal{A} \amalg \{ \,|\, \}$ obtained by adjoining the new letter $|$ to $\mathcal{A}$. However, multiplication of phrases is different from multiplication of words in this extended alphabet due to the additional symbol $|$ between w_k and x_1 in the formula above.

4.2. **Comultiplication in $\mathcal{P}$.** Fix a stable set of words L. We define here a comultiplication $\Delta = \Delta_L$ in $\mathcal{P}$. We begin with preliminary definitions.

Let w be a word of length $m \geq 1$. A *cut* of w of length $k \geq 0$ is a sequence of $2k$ integers $1 \leq i_1 < j_1 < i_2 < j_2 < ... < i_k < j_k \leq m+1$ such that the k words $w_{i_1,j_1}, ..., w_{i_k,j_k}$ belong to L. For $k = 1$, we additionally require that $(i_1, j_1) \neq (1, m+1)$. A cut of length 1 is nothing but a simple cut in terminology of Section 3.3. Every word has a unique *empty cut* $\emptyset$ of length 0.

To indicate that $c = (i_1, j_1, ..., i_k, j_k)$ is a cut of w we write $c \ll w$. We also write $\#c = \{1, 2, ..., k\}$. For $u \in \#c$, set $w^c_u = w_{i_u, j_u}$. The factors $w^c_1, ..., w^c_k$ of w are called *c-factors*. Finally we define a phrase

$$l_c(w) = \prod_{u \in \#c} (w^c_u) = (w^c_1 \,|\, w^c_2 \,|\, ... \,|\, w^c_k)$$

and a word

$$r_c(w) = w_{1,i_1} w_{j_1,i_2} w_{j_2,i_3} ... w_{j_{k-1},i_k} w_{j_k,m+1}.$$

To specify a cut of w of length k it is enough to specify k non-overlapping proper factors of w belonging to L and such that consecutive factors are separated by at least one letter. These factors form the phrase $l_c(w)$. Deleting them from w we obtain the word $r_c(w)$.

Set

$$\Delta(w) = w \otimes 1 + \sum_{c \ll w} l_c(w) \otimes r_c(w)$$

where c runs over all cuts of w. Note that the term corresponding to the empty cut is $1 \otimes w$. The mapping $w \mapsto \Delta(w)$ extends uniquely to an algebra homomorphism $\Delta = \Delta_L : \mathcal{P} \to \mathcal{P}^{\otimes 2}$.

For example, for a word w of length 1, we have $\Delta(w) = w \otimes 1 + 1 \otimes w$. If L is the set of all non-empty words and $A, B, C \in \mathcal{A}$, then

$$\Delta(AB) = AB \otimes 1 + 1 \otimes AB + A \otimes B + B \otimes A,$$

$$\Delta(ABC) = ABC \otimes 1 + 1 \otimes ABC + A \otimes BC + B \otimes AC + C \otimes AB + AB \otimes C + BC \otimes A + (A \,|\, C) \otimes B,$$

$$\Delta(AB \,|\, C) = \Delta(AB)\Delta(C) = (AB \,|\, C) \otimes 1 + C \otimes AB + (A \,|\, C) \otimes B + (B \,|\, C) \otimes A$$
$$+ AB \otimes C + 1 \otimes (AB \,|\, C) + A \otimes (B \,|\, C) + B \otimes (A \,|\, C).$$

Theorem 4.2.1. *The pair $(\mathcal{P}, \Delta)$ is a left-handed graded bialgebra with leading term ρ_L.*

Proof. The left-handedness and the claim concerning the leading term follow directly from the definitions. The only non-obvious assertion is the coassociativity of Δ. It suffices to prove that $(\mathrm{id} \otimes \Delta)\Delta(w) = (\Delta \otimes \mathrm{id})\Delta(w)$ for any word w. Set

$$\Theta(w) = \Delta(w) - w \otimes 1 = \sum_{c \ll w} l_c(w) \otimes r_c(w).$$

The mapping $w \mapsto \Theta(w)$ defines an R-linear homomorphism $\Theta : \mathcal{V} \to \mathcal{P} \otimes \mathcal{V}$. We have

$$(\Delta \otimes \mathrm{id})\Delta(w) = (\Delta \otimes \mathrm{id})\Theta(w) + \Delta(w) \otimes 1 = (\Delta \otimes \mathrm{id})\Theta(w) + \Theta(w) \otimes 1 + w \otimes 1 \otimes 1.$$

Similarly,

$$(\mathrm{id} \otimes \Delta)\Delta(w) = (\mathrm{id} \otimes \Delta)\Theta(w) + (\mathrm{id} \otimes \Delta)(w \otimes 1) = (\mathrm{id} \otimes \Theta)\Theta(w) + \Theta(w) \otimes 1 + w \otimes 1 \otimes 1.$$

Comparing these expressions we conclude that it is enough to prove that $(\Delta \otimes \mathrm{id})\Theta(w) = (\mathrm{id} \otimes \Theta)\Theta(w)$. If follows from the definitions that

$$(\mathrm{id} \otimes \Theta)\Theta(w) = \sum_{c \ll w} \sum_{e \ll r_c(w)} l_c(w) \otimes l_e(r_c(w)) \otimes r_e(r_c(w)).$$

We compute the right-hand side as follows.

For integers $i < j$, set $[i, j) = \{s \in \mathbb{Z} \,|\, i \leq s < j\}$. For cuts $d = (i_1, j_1, ..., i_k, j_k), d' = (i'_1, j'_1, ..., i'_{k'}, j'_{k'})$ of w, write $d \subset d'$ if $\cup_{u=1}^k [i_u, j_u) \subset \cup_{v=1}^{k'} [i'_v, j'_v)$. Suppose that $d \subset d'$. We say that the index $v \in \{1, ..., k'\}$ is *special* if $i'_v = i_u$ and $j'_v = j_u$ for some $u = 1, ..., k$. If v is non-special, then clearly $[i'_v, j'_v) \not\subseteq \cup_{u=1}^k [i_u, j_u)$. We can obtain a cut $\tilde{d}'$ of $r_d(w)$ by deleting all d-factors from the d'-factors of w numerated by non-special indices. That the $\tilde{d}'$-factors of $r_d(w)$ belong to L follows from the stability of L. The cut $\tilde{d}'$ is empty iff $d = d'$. Moreover, the formula $(d, d') \mapsto (d, \tilde{d}')$ establishes a bijective correspondence between pairs $(d \ll w, d' \ll w)$ with $d \subset d'$ and pairs $(c \ll w, e \ll r_c(w))$. Therefore

$$(\mathrm{id} \otimes \Theta)\Theta(w) = \sum_{d \ll w, d' \ll w, d \subset d'} l_d(w) \otimes l_{\tilde{d}'}(r_d(w)) \otimes r_{\tilde{d}'}(r_d(w)) = \sum_{d \ll w, d' \ll w, d \subset d'} l_d(w) \otimes l_{\tilde{d}'}(r_d(w)) \otimes r_{d'}(w).$$

We claim that the right-hand side is equal to $(\Delta \otimes \mathrm{id})\Theta(w)$. (The proof of this does not use the stability of L). If follows from the definitions that

$$(\Delta \otimes \mathrm{id})\Theta(w) = \sum_{c \ll w} \prod_{u \in \#c} \Big(w^c_u \otimes 1 + \sum_{e_u \ll w^c_u} l_{e_u}(w^c_u) \otimes r_{e_u}(w^c_u) \Big) \otimes r_c(w)$$

$$= \sum_{c \ll w} \sum_{I \subset \# c} \sum_{\{e_u \ll w_u^c\}_{u \in \# c - I}} \Big(\prod_{v \in I} w_v^c \prod_{u \in \# c - I} l_{e_u}(w_u^c) \Big) \otimes \prod_{u \in \# c - I} r_{e_u}(w_u^c) \otimes r_c(w).$$

Here all the products are ordered in accordance with the natural order in $\# c$. For example, if $\# c = \{1,2,3\}$ and $I = \{1,3\}$, then the term in the big round brackets on the right-hand side is $w_1^c \, l_{e_2}(w_2^c) \, w_3^c$.

Given a cut $c = (i_1, j_1, ..., i_k, j_k)$ of w and $u \in \# c$, every cut e_u of w_u^c yields a cut $\hat{e}_u$ of w by adding $i_u - 1$ to all terms of e_u. A set $I \subset \# c$ gives rise to a cut $\hat{I}$ of w formed by the indices $\{i_v, j_v\}_{v \in I}$. With a tuple $(c \ll w, I \subset \# c, \{e_u \ll w_u^c\}_{u \in \# c - I})$ we associate a pair (d, d') where $d' = c$ and d is the cut of w obtained as the union of $\hat{I}$ with all $\{\hat{e}_u\}_{u \in \# c - I}$. This establishes a bijective correspondence between such tuples and the pairs $(d \ll w, d' \ll w)$ with $d \subset d'$. The corresponding terms in the expansions of $(\mathrm{id} \otimes \Theta)\Theta(w)$ and $(\Delta \otimes \mathrm{id})\Theta(w)$ are equal. Therefore $(\mathrm{id} \otimes \Theta)\Theta(w) = (\Delta \otimes \mathrm{id})\Theta(w)$. □

Corollary 4.2.2. *The bialgebra $(\mathcal{P}, \Delta)$ is a Hopf algebra.*

Proof. The augmentation $\varepsilon : \mathcal{P} \to R$ sending all non-empty phrases to 0 and sending 1 to 1 is a counit of Δ. It remains to show the existence of an antipode. Consider the unique R-linear endomorphism s of $\mathcal{P}$ such that $s(1) = 1$, $s(ab) = s(b)s(a)$ for all $a, b \in \mathcal{P}$ and the value of s on words is defined by induction on the length as follows: for a word w of length 1, set $s(w) = -w$; for a word w of length ≥ 2, set

$$s(w) = -w - \sum_{c \ll w, c \neq \emptyset} l_c(w) \, s(r_c(w)) \in \mathcal{P}$$

where we use that $r_c(w)$ is shorter than w. These formulas imply that $\mu(\mathrm{id} \otimes s)\Delta(w) = \varepsilon(w)$ where μ is multiplication in $\mathcal{P}$. In other words, s is a right inverse of $\mathrm{id} : \mathcal{P} \to \mathcal{P}$ with respect to the (associative) convolution product $\bullet$ in $\mathrm{End}_R(\mathcal{P})$ defined by $f \bullet g = \mu(f \otimes g)\Delta$ for $f, g \in \mathrm{End}_R(\mathcal{P})$. Similar inductive formulas show that id has a left inverse $s' \in \mathrm{End}_R(\mathcal{P})$ and then $s' = s' \bullet (\mathrm{id} \bullet s) = (s' \bullet \mathrm{id}) \bullet s = s$. Therefore s is an antipode for $(\mathcal{P}, \Delta)$. □

4.3. **Phrase indicators.** By *phrase indicators* we mean R-linear homomorphisms $\mathcal{P} \to R$. By the general theory of bialgebras, the comultiplication Δ_L in $\mathcal{P}$ induces an associative multiplication $\circ_L$ in the module of phrase indicators $\mathcal{P}^* = \mathrm{Hom}_R(\mathcal{P}, R)$. The product $f \circ_L g$ may distinguish phrases indistinguishable by $f, g \in \mathcal{P}^*$. For example, let L consist of all non-empty words and let ℓ and f_B be the phrase indicators counting the number of words in a phrase and the number of appearances of $B \in \mathcal{A}$ in a phrase, respectively. Then the values of $\ell \circ_L f_B$ on the 1-word phrases (ABC) and (ACB) (where A, B, C are distinct letters in $\mathcal{A}$) are 4 and 3, respectively.

The additive homomorphism $\Theta : \mathcal{V} \to \mathcal{P} \otimes \mathcal{V}$ constructed in the proof of Theorem 4.2.1 makes $\mathcal{V}$ into a comodule over the bialgebra $(\mathcal{P}, \Delta)$. This induces a right action of the algebra $\mathcal{P}^*$ on $\mathcal{V}$. Using the antiautomorphism of $\mathcal{P}^*$ induced by the antipode in $\mathcal{P}$, we can transform the right action of $\mathcal{P}^*$ into a left action. The right and left actions of a phrase indicator $f \in \mathcal{P}^*$ on $\mathcal{V}$ depend only on the values of f on phrases with all words in L. This can be formalized as follows. Let $\mathcal{P}_L$ be the subalgebra of $\mathcal{P}$ additively generated by the phrases whose all words belong to L (including the empty phrase 1). It is clear that $\Delta_L(\mathcal{P}_L) \subset \mathcal{P}_L \otimes \mathcal{P}_L$ so that $\mathcal{P}_L$ is a Hopf subalgebra of $\mathcal{P}$. Clearly, $\Theta(\mathcal{V}) \subset \mathcal{P}_L \otimes \mathcal{V}$. In this way $\mathcal{V}$ acquires the structure of a comodule over $\mathcal{P}_L$. The actions of the algebras $\mathcal{P}^*, (\mathcal{P}_L)^*$ on $\mathcal{P}$ are compatible via the algebra homomorphism $\mathcal{P}^* \to (\mathcal{P}_L)^*$ induced by the inclusion $\mathcal{P}_L \subset \mathcal{P}$.

4.4. **Dual Hopf algebra.** We can define a Hopf algebra dual to $\mathcal{P}$. Consider the algebra $\mathcal{P}^*$ with multiplication induced by Δ_L and quasi-comultiplication induced by multiplication in $\mathcal{P}$. Consider the embedding $\delta : \mathcal{P} \hookrightarrow \mathcal{P}^*$ mapping a phrase p into the phrase indicator δ_p whose value on p is 1 and whose value on all other phrases is 0. It is easy to see that $\delta(\mathcal{P})$ is a subalgebra of $\mathcal{P}^*$. In this way the module $\mathcal{P}$ acquires a new associative multiplication $\circ_L$. The quasi-comultiplication in $\mathcal{P}^*$ induces a genuine comultiplication in $\mathcal{P}$ transforming a phrase $(w_1 \,|\, ... \,|\, w_k)$ into $\sum_{i=0}^{k} (w_1 \,|\, ... \,|\, w_i) \otimes (w_{i+1} \,|\, ... \,|\, w_k)$. This makes the algebra $(\mathcal{P}, \circ_L)$ into a Hopf algebra. By its very definition, it is dual to $(\mathcal{P}, \Delta_L)$.

For completeness, we describe multiplication $\circ_L$ in $\mathcal{P}$ explicitly. For a phrase $p = (w_1 \,|\, ... \,|\, w_k)$ and a non-empty word y, set $p * y = 0$ if at least one of the words $w_1, ..., w_k$ does not belong to L. If $w_1, ..., w_k \in L$, set

$$p * y = \sum_{y = x_1 \cdots x_{k+1}} x_1 w_1 x_2 w_2 ... w_k x_{k+1} \in \mathcal{V} \subset \mathcal{P}$$

where the sum runs over all sequences of words $x_1,...,x_{k+1}$ (some of them possibly empty) such that $y = x_1 \cdots x_{k+1}$. In particular if $p=1$ (that is if $k=0$), then $p*y=y$. For empty word y, set $p*y=0$ if $k\neq 1$ and $p*y=w_1$ if $k=1$.

Given a non-empty phrase $p=(w_1\,|\,...\,|\,w_k)$, denote by $S(p)$ the set of all finite sequences of phrases $p_1,...,p_t$ (some of them possibly empty) such that $p=p_1p_2...p_t$. Denote by $W(p)$ the set of all finite sequences of words obtained by inserting empty words in the sequence $w_1, w_2, ..., w_k$.

It is easy to verify that $1\circ_L p = p\circ_L 1 = p$ for all $p\in\mathcal{P}$. For non-empty phrases p,q,

$$p\circ_L q = \delta^{-1}(\delta_p\delta_q) = \sum_{t\geq 1}\ \sum_{(p_1,...,p_t)\in S(p),(y_1,...,y_t)\in W(q)}\ \prod_{i=1}^{t} p_i * y_i \in \mathcal{P}.$$

The right-hand side contains only a finite number of non-zero terms because $p_i*y_i=0$ unless p_i and/or y_i are non-empty.

4.5. **Example.** Let the alphabet $\mathcal{A}$ consist of one letter A. Then $\mathcal{P}$ is a free associative (non-commutative) unital algebra over R freely generated by the words $A, A^2=AA, A^3=AAA, ...$ (caution: A^2 is not the square of A in $\mathcal{P}$). The comultiplication Δ_L in $\mathcal{P}$ corresponding to $L=\{A^m\}_{m\geq 1}$ is computed by

$$\Delta_L(A^m) = A^m\otimes 1 + 1\otimes A^m + \sum_{m>m_1\geq 1} (m+1-m_1)\, A^{m_1}\otimes A^{m-m_1}$$

$$+\sum_{k\geq 2}\ \sum_{m_1,...,m_k\geq 1, m_1+...+m_k\leq m-k+1} \binom{m+1-m_1-...-m_k}{k} A^{m_1}\cdots A^{m_k}\otimes A^{m-m_1-...-m_k}.$$

If L consists of the words whose length is divisible by a given integer $N\geq 1$, then the formula for Δ_L is the same with the restriction that $m_1,...,m_k$ are divisible by N.

4.6. **Functoriality.** Any mapping α from an alphabet $\mathcal{A}$ to an alphabet $\mathcal{A}'$ extends to words letter-wise. Denote the resulting mapping by $\tilde{\alpha}$. A stable set of words L' in the alphabet $\mathcal{A}'$ gives rise to a stable set of words $L=\tilde{\alpha}^{-1}(L')$ in the alphabet $\mathcal{A}$. It is clear that $\tilde{\alpha}$ induces an R-homomorphism $(\mathcal{V}(\mathcal{A}),\rho_L)\to(\mathcal{V}(\mathcal{A}'),\rho_{L'})$ of pre-Lie coalgebras. The latter extends by multiplicativity to a homomorphism $(\mathcal{P}(\mathcal{A}),\Delta_L)\to(\mathcal{P}(\mathcal{A}'),\Delta_{L'})$ of Hopf algebras. For example, if L is the set of all non-empty words in the alphabet $\mathcal{A}$, then any permutation of $\mathcal{A}$ induces an automorphism of the pre-Lie coalgebra $(\mathcal{V}(\mathcal{A}),\rho_L)$ and an automorphism of the Hopf algebra $(\mathcal{P}(\mathcal{A}),\Delta_L)$.

5. Coalgebra of words: second construction

Fix from now on a mapping $\mu:\mathcal{A}\times\mathcal{A}\to R$. We derive from μ a pre-Lie comultiplication $\rho=\rho_\mu$ in the R-module $\mathcal{W}=\mathcal{V}\oplus R\phi$.

5.1. **Simple inscriptions.** Let w be a word of length $m\geq 0$. A *simple inscription* in w is a pair $i,j\in\{1,2,...,m\}$ with $i<j$. To indicate that $a=(i,j)$ is a simple inscription in w we write $a\dagger w$. Consider the word $l_a(w)=w_{i+1,j}$ of length $j-i-1$ and the word $r_a(w)=w_{1,i}w_{j+1,m+1}$ of length $m+i-j-1$. Set $\mu(w|_a)=\mu(w(i),w(j))\in R$. For example, if $w=ABABB$ and $a=\{1,4\}$, then $r_a(w)=B$, $l_a(w)=BA$, and $\mu(w|_a)=\mu(A,B)$.

Set

$$\rho(w)=\sum_{a\dagger w}\mu(w|_a)\, l_a(w)\otimes r_a(w)\in\mathcal{W}\otimes\mathcal{W}$$

where a runs over all simple inscriptions in w. If w is an empty word or a 1-letter word, then w has no simple inscriptions so that $\rho(w)=0$. Extending ρ by linearity, we obtain a comultiplication $\rho:\mathcal{W}\to\mathcal{W}\otimes\mathcal{W}$.

Theorem 5.1.1. *ρ is a pre-Lie comultiplication in $\mathcal{W}$.*

Proof. Consider a word w of length m and two simple inscriptions $a=(i,j)$ and $b=(i',j')$ in w. We write $a<b$ if $j<i'$. In this case set

$$w(a,b)=\mu(w|_a)\,\mu(w|_b)\, l_b(w)\otimes l_a(w)\otimes w_{1,i}w_{j+1,i'}w_{j'+1,m+1}\in\mathcal{W}^{\otimes 3}.$$

We write $a>b$ if $i>j'$ and set then

$$w(a,b)=\mu(w|_a)\,\mu(w|_b)\, l_b(w)\otimes l_a(w)\otimes w_{1,i'}w_{j'+1,i}w_{j+1,m+1}\in\mathcal{W}^{\otimes 3}.$$

Note that $a > b$ iff $b < a$ and then $w(a,b) = P^{1,2}(w(b,a))$. We write $a \sqsubset b$ if $i < i' < j' < j$. Set then

$$w(a,b) = \mu(w|_a)\,\mu(w|_b)\,l_b(w) \otimes w_{i+1,i'} w_{j'+1,j} \otimes r_a(w) \in \mathcal{W}^{\otimes 3}.$$

We expand $(\mathrm{id} \otimes \rho)\rho(w)$ from definitions:

$$(\mathrm{id} \otimes \rho)\rho(w) = \sum_{b \dagger w} \mu(w|_b)\,l_b(w) \otimes \rho(r_b(w))$$

$$= \sum_{b \dagger w} \mu(w|_b) \sum_{a \dagger r_b(w)} \mu(r_b(w)|_a)\,l_b(w) \otimes l_a(r_b(w)) \otimes r_a(r_b(w)).$$

For $b = (i', j')\dagger w$, we can describe all simple inscriptions in $r_b(w) = w_{1,i'} w_{j'+1,m+1}$ as follows. A simple inscription $a \dagger w$ such that $a < b$ is automatically a simple inscription in $r_b(w)$. A simple inscription $a = (i,j)$ in w such that $a > b$ yields a simple inscription in $r_b(w)$ by subtracting $j' + 1 - i'$ from both i and j. A simple inscription $a = (i,j)$ in w such that $a \sqsubset b$ yields a simple inscription $(i, j - (j' + 1 - i'))$ in $r_b(w)$. It is clear that every simple inscription in $r_b(w)$ arises in exactly one of these 3 ways from a certain $a \dagger w$. The corresponding term in the expansion of $(\mathrm{id} \otimes \rho)\rho(w)$ is $w(a,b)$. Therefore $(\mathrm{id} \otimes \rho)\rho(w) = x + y + z$ where

$$x = \sum_{a,b \dagger w, a<b} w(a,b), \quad y = \sum_{a,b \dagger w, a>b} w(a,b), \quad z = \sum_{a,b \dagger w, a \sqsubset b} w(a,b).$$

A similar (in fact easier) computation shows that $(\rho \otimes \mathrm{id})\rho(w) = z$. Therefore

$$\tilde{\rho}(w) = ((\mathrm{id} \otimes \rho)\rho - (\rho \otimes \mathrm{id})\rho)(w) = x + y.$$

By the remarks above, $y = P^{1,2}(x)$ so that $\tilde{\rho}(w) = x + y$ is invariant under $P^{1,2}$. □

5.2. **Extended word indicators.** By the general theory exposed in Section 2, the pre-Lie comultiplication ρ_μ induces a pre-Lie multiplication $\star_\mu$ and a Lie bracket $[f,g]_\mu = f \star_\mu g - g \star_\mu f$ in the module $\mathcal{W}^* = \mathrm{Hom}_R(\mathcal{W}, R)$. The elements of $\mathcal{W}^*$ are called *extended word indicators*. The module $\mathcal{W}^*$ admits a decreasing filtration $\mathcal{W}^* = \mathcal{W}^{*(0)} \supset \mathcal{W}^{*(1)} \supset ...$ where $\mathcal{W}^{*(m)}$ consists of the indicators annihilating all words of length $\leq m-1$. It is clear that $\mathcal{W}^{*(m)} \star_\mu \mathcal{W}^{*(n)} \subset \mathcal{W}^{*(m+n+2)}$ and $[\mathcal{W}^{*(m)}, \mathcal{W}^{*(n)}]_\mu \subset \mathcal{W}^{*(m+n+2)}$ for all m, n. This implies that $\mathcal{W}^* = \operatorname{proj\,lim}_m (\mathcal{W}^*/\mathcal{W}^{*(m)})$ is a projective limit of nilpotent Lie algebras.

Recall the Lie algebra action of $\mathcal{W}^*$ on $\mathcal{W}$ induced by ρ_μ. For $f \in \mathcal{W}^*$ and a word w,

$$fw = -\sum_{a \dagger w} \langle l_a(w), f \rangle\, \mu(w|_a)\, r_a(w) \in \mathcal{W}.$$

Consider the filtration $0 = \mathcal{W}_{-1} \subset R\emptyset = \mathcal{W}_0 \subset \mathcal{W}_1 \subset ...$ of $\mathcal{W}$ where $\mathcal{W}_m$ is generated by the words of length $\leq m$. It is clear that $f\mathcal{W}_m \subset \mathcal{W}_{m-2}$ for all m and all $f \in \mathcal{W}^*$. This implies that the action of $\mathcal{W}^*$ on $\mathcal{W}_m$ is nilpotent for all m.

Consider the embedding $\delta : \mathcal{W} \hookrightarrow \mathcal{W}^*$ mapping a word w into the extended word indicator δ_w whose value on w is 1 and whose value on all other words is 0. If $\mathcal{A}$ is finite or more generally if μ takes non-zero values only on a finite subset of $\mathcal{A} \times \mathcal{A}$, then the image of δ is closed under $\star_\mu$. This induces a pre-Lie multiplication on $\mathcal{W}$. We leave it to the reader to give an explicit formula for it.

If $R \supset \mathbb{Q}$, then the Campbell-Baker-Hausdorff formula defines a group multiplication in $\mathcal{W}^*$ as in Section 3.6. The resulting pro-nilpotent group $\operatorname{Exp}\mathcal{W}^*$ is the "Lie group" with Lie algebra $\mathcal{W}^*$. The Lie algebra action of $\mathcal{W}^*$ on $\mathcal{W}$ induced by ρ integrates to a group action of $\operatorname{Exp}\mathcal{W}^*$ on $\mathcal{W}$ as in Section 3.6.

5.3. **Example.** Let $\mu : \mathcal{A} \times \mathcal{A} \to R$ send a pair (A, B) to 1 if $A = B$ and to 0 if $A \neq B$. Let $w = ABACBA$ with $A, B, C \in \mathcal{A}$. Then

$$\rho_\mu(w) = B \otimes CBA + CB \otimes AB + BACB \otimes \phi + AC \otimes AA.$$

For any word indicator f,

$$fw = -\langle B, f \rangle\, CBA - \langle CB, f \rangle\, AB - \langle BACB, f \rangle\, \phi - \langle AC, f \rangle\, AA.$$

For example, let the indicator f_A compute the total number of occurencies of the letter A in a word. Then $f_A w = -\phi - AA$, $f_B w = -CBA - AB - 2\phi$, and $f_C w = -AB - \phi - AA$. If ℓ is the indicator computing the length of a word, then $\ell w = -CBA - 2AB - 4\phi - 2AA$.

6. Hopf algebra of phrases: second construction

6.1. **Comultiplication Δ_μ.** Recall the algebra of phrases $\mathcal{Q}$ defined in Section 4.1. The inclusion $\mathcal{W} \hookrightarrow \mathcal{Q}$ as 1-word phrases extends to an isomorphism of the tensor algebra $T(\mathcal{W})$ onto $\mathcal{Q}$. We shall identify $T(\mathcal{W})$ with $\mathcal{Q}$. The results above suggest that there may exist a left-handed comultiplication in $\mathcal{Q}$ with leading term ρ_μ. We define such a comultiplication $\Delta = \Delta_\mu$ in $\mathcal{Q}$.

Let w be a word of length $m \geq 0$. By an *inscription* in w we shall mean a subword of w of even length. More precisely, an inscription in w of length $k \geq 0$ is a set $\alpha \subset \{1, 2, ..., m\}$ consisting of $2k$ elements. We shall list these elements in the increasing order and write $\alpha = (i_1, j_1, ..., i_k, j_k)$ where $i_1 < j_1 < ... < i_k < j_k$. Every word has a unique *empty inscription* of length 0. To indicate that $\alpha = (i_1, j_1, ..., i_k, j_k)$ is an inscription in w we write $\alpha \ddagger w$. Set $\#\alpha = \{1, 2, ..., k\}$ and $supp(\alpha) = \cup_{u\in\#\alpha}[i_u, j_u]$ where $[i, j] = \{k \in \mathbb{Z} \,|\, i \leq k \leq j\}$. With each $u \in \#\alpha$, we associate the word $w_u^\alpha = w_{i_u+1, j_u}$. It is empty iff $i_u + 1 = j_u$. We define a phrase

$$l_\alpha(w) = \prod_{u\in\#\alpha} w_u^\alpha = (w_1^\alpha \,|\, w_2^\alpha \,|\, ... \,|\, w_k^\alpha) \in \mathcal{Q}.$$

Clearly $l_\alpha(w) = 1$ iff α is void. We also associate with α an element of the ground ring

$$\mu(w, \alpha) = \prod_{u\in\#\alpha} \mu(w(i_u), w(j_u))$$

and a word

$$r_\alpha(w) = w_{1,i_1} w_{j_1+1,i_2} \cdots w_{j_{k-1}+1,i_k} w_{j_k+1,m+1}.$$

If $\alpha = \emptyset$, then $l_\alpha(w) = 1$, $\mu(w, \alpha) = 1$, and $r_\alpha(w) = w$.

Set

$$\Delta(w) = w \otimes 1 + \sum_{\alpha\ddagger w} \mu(w, \alpha)\, l_\alpha(w) \otimes r_\alpha(w)$$

where α runs over all inscriptions in w. Note that the term corresponding to $\alpha = \emptyset$ is $1 \otimes w$.

The mapping $w \mapsto \Delta(w)$ extends uniquely to an algebra homomorphism $\Delta : \mathcal{Q} \to \mathcal{Q}\otimes\mathcal{Q}$.

Theorem 6.1.1. *The pair $(\mathcal{Q}, \Delta)$ is a left-handed graded bialgebra with leading term ρ_μ.*

Proof. The only non-obvious assertion is the coassociativity of Δ. It suffices to prove that $(\mathrm{id} \otimes\Delta)\Delta(w) = (\Delta \otimes \mathrm{id})\Delta(w)$ for any word w. Set

$$\Theta(w) = \Delta(w) - w \otimes 1 = \sum_{\alpha\ddagger w} \mu(w, \alpha)\, l_\alpha(w) \otimes r_\alpha(w).$$

The mapping $w \mapsto \Theta(w)$ defines an R-linear homomorphism $\Theta : \mathcal{W} \to \mathcal{P} \otimes \mathcal{W}$. A computation similar to the one in the proof of Theorem 4.2.1 shows that it is enough to prove that $(\Delta \otimes \mathrm{id})\Theta(w) = (\mathrm{id} \otimes\Theta)\Theta(w)$.

If w has length 0 or 1, then $\Theta(w) = 1 \otimes w$ and $(\Delta \otimes \mathrm{id})\Theta(w) = 1 \otimes 1 \otimes w = (\mathrm{id} \otimes\Theta)\Theta(w)$. Suppose from now on that w has length ≥ 2. For inscriptions β and η in w, we write $\beta \lhd \eta$ if $supp(\beta) \cap \eta = \emptyset$. Striking out from w all letters numerated by elements of the set $supp(\beta)$ we obtain the word $r_\beta(w)$. If $\beta \lhd \eta$, then the letters of w numerated by elements of η survive in $r_\beta(w)$ and form an inscription in $r_\beta(w)$ denoted η/β. The formula $(\beta, \eta) \mapsto (\beta, \eta/\beta)$ establishes a bijective correspondence between pairs $(\beta\ddagger w, \eta\ddagger w)$ such that $\beta \lhd \eta$ and pairs $(\alpha\ddagger w, \gamma\ddagger r_\alpha(w))$. Therefore

$$(\mathrm{id} \otimes\Theta)\Theta(w) = \sum_{\alpha\ddagger w} \mu(w, \alpha) \sum_{\gamma\ddagger r_\alpha(w)} \mu(r_\alpha(w), \gamma)\, l_\alpha(w) \otimes l_\gamma(r_\alpha(w)) \otimes r_\gamma(r_\alpha(w))$$

$$= \sum_{\beta,\eta\ddagger w, \beta\lhd\eta} \mu(w, \beta)\, \mu(w, \eta)\, l_\beta(w) \otimes l_{\eta/\beta}(r_\beta(w)) \otimes r_{\eta/\beta}(r_\beta(w)).$$

On the other hand,

$$(\Delta \otimes \mathrm{id})\Theta(w) = \sum_{\alpha\ddagger w} \mu(w, \alpha)\, \Delta(l_\alpha(w)) \otimes r_\alpha(w)$$

$$= \sum_{\alpha\ddagger w} \mu(w, \alpha) \prod_{u\in\#\alpha} \Big(w_u^\alpha \otimes 1 + \sum_{\varepsilon_u\ddagger w_u^\alpha} \mu(w_u^\alpha, \varepsilon_u)\; l_{\varepsilon_u}(w_u^\alpha) \otimes r_{\varepsilon_u}(w_u^\alpha)\Big) \otimes r_\alpha(w)$$

$$= \sum_{\alpha\ddagger w} \mu(w, \alpha) \sum_{I\subset\#\alpha} \sum_{\{\varepsilon_u\ddagger w_u^\alpha\}_{u\in\#\alpha-I}} \prod_{u\in I} w_u^\alpha \prod_{u\in\#\alpha-I} \mu(w_u^\alpha, \varepsilon_u)\; l_{\varepsilon_u}(w_u^\alpha) \otimes \prod_{u\in\#\alpha-I} r_{\varepsilon_u}(w_u^\alpha) \otimes r_\alpha(w).$$

Here the products are ordered in accordance with the order of the indices in $\#\alpha$. For example, if $\#\alpha = \{1,2,3\}$ and $I = \{1,3\}$, then the first tensor factor on the right hand side is $\mu(w_2^\alpha, \varepsilon_2)\, w_1^\alpha\, l_{\varepsilon_2}(w_2^\alpha)\, w_3^\alpha$.

Consider an inscription $\alpha = (i_1, j_1, ..., i_k, j_k)$ in w. Given $u \in \#\alpha$ and an inscription ε_u in w_u^α, we obtain an inscription (of the same length) $\hat{\varepsilon}_u$ in w by adding i_u to all terms of ε_u. With a tuple $(\alpha \ddagger w, I \subset \#\alpha, \{\varepsilon_u \ddagger w_u^\alpha\}_{u \in \#\alpha - I})$ we associate two inscriptions β, η in w by $\beta = \{i_u, j_u\}_{u\in I} \cup \cup_{u \in \#\alpha - I} \hat{\varepsilon}_u$ and $\eta = \{i_u, j_u\}_{u \in \#\alpha - I}$. This defines a bijective correspondence between such tuples and the pairs $(\beta \ddagger w, \eta \ddagger w)$ such that $\beta \triangleleft \eta$. The corresponding terms in the expansions for $(\Delta \otimes \mathrm{id})\Theta(p)$ and $(\mathrm{id} \otimes \Theta)\Theta(p)$ are equal. Therefore $(\Delta \otimes \mathrm{id})\Theta(p) = (\mathrm{id} \otimes \Theta)\Theta(p)$. □

Corollary 6.1.2. *The algebra $\mathcal{Q}$ with comultiplication Δ is a Hopf algebra.*

Proof. The augmentation $\mathcal{Q} \to R$ sending all non-empty phrases to 0 and sending the empty phrase to 1 is a counit of Δ. The existence of an antipode is shown as in the proof of Corollary 4.2.2. □

6.2. **Extended phrase indicators.** Homomorphisms $\mathcal{Q} \to R$ are called *extended phrase indicators.* The comultiplication Δ_μ in $\mathcal{Q}$ induces an associative multiplication $\circ_\mu$ in $\mathcal{Q}^* = \mathrm{Hom}_R(\mathcal{Q}, R)$. It is easy to give examples showing that $f \circ_\mu g$ may distinguish phrases indistinguishable by $f, g \in \mathcal{Q}^*$.

The additive homomorphism $\Theta : \mathcal{W} \to \mathcal{Q} \otimes \mathcal{W}$ constructed in the proof of Theorem 6.1.1 makes $\mathcal{W}$ into a comodule over the bialgebra $(\mathcal{Q}, \Delta)$. The leading term of Θ is the pre-Lie comultiplication ρ_μ in $\mathcal{W}$. The coaction Θ induces a right action of the algebra of extended phrase indicators $\mathcal{Q}^*$ on $\mathcal{W}$. Using the antiautomorphism of $\mathcal{Q}^*$ induced by the antipode in $\mathcal{Q}$, we can transform the right action of $\mathcal{Q}^*$ into a left action.

6.3. **Dual Hopf algebra.** If $\mathcal{A}$ is finite or more generally if μ takes non-zero values only on a finite subset of $\mathcal{A} \times \mathcal{A}$, then a Hopf algebra dual to $\mathcal{Q}$ can be constructed as follows. Consider the algebra $\mathcal{Q}^*$ with multiplication induced by Δ_μ and quasi-comultiplication induced by multiplication in $\mathcal{Q}$. Consider the embedding $\delta : \mathcal{Q} \hookrightarrow \mathcal{Q}^*$ mapping a phrase p into the phrase indicator δ_p whose value on p is 1 and whose value on all other phrases is 0. Under our assumptions on μ, $\delta(\mathcal{Q})$ is a subalgebra of $\mathcal{Q}^*$. This induces a new associative multiplication $\circ_\mu$ in $\mathcal{Q}$. The quasi-comultiplication in $\mathcal{Q}^*$ induces a genuine comultiplication in $\mathcal{Q}$ transforming a phrase $(w_1 \,|\, ... \,|\, w_k)$ into $\sum_{i=0}^k (w_1 \,|\, ... \,|\, w_i) \otimes (w_{i+1} \,|\, ... \,|\, w_k)$. This makes the algebra $(\mathcal{Q}, \circ_\mu)$ into a Hopf algebra. By its very definition, it is dual to $(\mathcal{Q}, \Delta_\mu)$.

6.4. **Independence of the basis.** Let M be the free R-module with basis $\mathcal{A}$. The mapping $\mu : \mathcal{A} \times \mathcal{A} \to R$ extends to a bilinear form $M \times M \to R$ also denoted μ. The constructions above produce a pre-Lie comultiplication ρ_μ in $\mathcal{W} = T(M)$ and a Hopf comultiplication Δ_μ in $\mathcal{Q} = T(\mathcal{W})$. An inspection of these constructions shows that they are entirely determined by the form $\mu : M \times M \to R$ and do not depend on the basis $\mathcal{A}$ in M. This version of the constructions applies to an arbitrary (not necessarily free) R-module M endowed with a bilinear form $\mu : M \times M \to R$. It yields a pre-Lie coalgebra $(T(M), \rho_\mu)$ and a Hopf algebra $(T(T(M)), \Delta_\mu)$.

A homomorphism of R-modules $\psi : M \to M'$ compatible with bilinear forms $\mu : M \times M \to R, \mu' : M' \times M' \to R$ (so that $\mu = \mu'(\psi \times \psi)$) induces a homomorphism of pre-Lie coalgebras $(T(M), \rho_\mu) \to (T(M'), \rho_{\mu'})$ and a homomorphism of Hopf algebras $(T(T(M)), \Delta_\mu) \to (T(T(M')), \Delta_{\mu'})$. In particular, μ-preserving automorphisms of M induce automorphisms of $(T(M), \rho_\mu)$ and of $(T(T(M)), \Delta_\mu)$.

6.5. **Relations with algebras of trees.** Let $\mu : \mathcal{A} \times \mathcal{A} \to R$ be the mapping sending a pair $(A, B) \in \mathcal{A} \times \mathcal{A}$ to 1 if $A = B$ and to 0 if $A \neq B$. We relate the Hopf algebra $(\mathcal{Q}, \Delta_\mu)$ with the Hopf algebra of planar rooted trees due to Connes-Kreimer [CK] and Foissy [Fo].

We say that a word w in the alphabet $\mathcal{A}$ is *unlaced* if (i) every letter of $\mathcal{A}$ either does not appear in w or appears in w twice and (ii) for any distinct letters A, B appearing in w the word $ABAB$ is not a subword of w. The latter condition may be reformulated by saying that w must have the form $\cdots A \cdots A \cdots B \cdots B \cdots$ or $\cdots A \cdots B \cdots B \cdots A \cdots$ or $\cdots B \cdots B \cdots A \cdots A \cdots$. Let $U \subset \mathcal{W}$ be the free R-module generated by the unlaced words (including the empty word). It follows from the definitions, that $\rho_\mu(U) \subset U \otimes U$. Similarly, the tensor algebra $T(U)$ is a Hopf subalgebra of $(\mathcal{Q} = T(\mathcal{W}), \Delta_\mu)$. Restricting Δ_μ to $T(U)$ we obtain a Hopf algebra $(T(U), \Delta_\mu)$.

Unlaced words can be described in terms of decorated planar rooted (finite) trees as follows. We say that a tree is *decorated* if every its edge is labeled by a letter of the alphabet $\mathcal{A}$ so that different edges are labeled by different letters. Each decorated planar rooted tree $\tau \subset \mathbb{R}^2$ gives rise to a word $w(\tau)$ as follows. Consider a narrow neighborhood V of τ in $\mathbb{R}^2$. If τ has n edges then the circle ∂V consists of $2n$ arcs going closely to

edges of τ. Starting at a point $x \in \partial V$ near the root of τ and moving along ∂V counterclockwise we write down the labels of the corresponding edges of τ until the first return to x. This gives $w(\tau)$. It is clear from the definitions that this word is unlaced. For example, if τ is a point, then $w(\tau) = \phi$. If τ consists of a single edge labeled with A, then $w(\tau) = AA$. If τ is a Y-shaped tree with 3 edges, then $w(\tau) = ABBCCA$ where A is the label of the edge incident to the root and B, C are the labels of the two other edges.

An induction on the number of edges shows that the formula $\tau \mapsto w(\tau)$ establishes a bijective correspondence between decorated planar rooted trees (considered up to ambient isotopy in the plane) and unlaced words. In this way the tensor algebra $T(U)$ can be identified with the free associative (non-commutative) algebra $\mathcal{H}$ generated by decorated planar rooted trees. A comparison of definitions shows that under this identification the comultiplication Δ_μ in $T(U)$ coincides with the Connes-Kreimer-Foissy comultiplication in $\mathcal{H}$. Note that Connes and Kreimer considered a commutative algebra generated by rooted trees (without planar structure). A non-commutative version of their definition was given by Foissy for planar rooted trees. In his paper, Foissy decorates vertices of trees rather than edges. However his definitions directly extend to trees with decorated edges.

References

[CK] A. Connes, D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys. 199 (1998), no. 1, 203–242.

[Fo] L. Foissy, *Les algèbres de Hopf des arbres enracinés décorés. I*. Bull. Sci. Math. 126 (2002), no. 3, 193–239.

[Ge] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) 78 (1963), 267–288.

[Tu] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. (4) 24 (1991), no. 6, 635–704.

[Vi] E.B. Vinberg, *The theory of convex homogeneous cones*, Trans. Mosc. Math. Soc. 12 (1963), 340-403; translation from Tr. Mosk. Mat. Ob-va 12 (1963), 303-358.

IRMA, Université Louis Pasteur - C.N.R.S.,
7 rue René Descartes
F-67084 Strasbourg
France
e-mail: turaev@math.u-strasbg.fr